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Directed compact percolation: cluster size and hyperscaling

J W Essam

Department of Mathematics, Royal Holloway and Bedford New College, Egham Hill, Egham, Surrey TW20 0EX, UK

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Abstract. Exact recurrence relations are obtained for the length and size distributions of compact directed percolation clusters on the square lattice. The corresponding relation for the moment generating function of the length distribution is obtained in closed form whereas in the case of the size distribution only the first three moments are obtained. The work is carried out for clusters grown from a seed of arbitrary width on an anisotropic lattice. A duality property is shown to exist which relates the moment generating functions on the two sides of the critical curve. The moments of both distributions have critical exponents which satisfy a *constant gap hypothesis* with gap exponents $\nu_{\Uparrow} = 2$ and $\Delta = 3$ corresponding to the scaling length and scaling size respectively. The usual hyperscaling relation for directed percolation is found to be invalid for compact clusters and is replaced by

$$\Delta = \nu_{\Uparrow} + D\nu_{\perp}$$

where $\nu_{\perp}$ is the exponent corresponding to the cluster width and D is the number of transverse dimensions ($=1$ for the square lattice).

1. Introduction

Consider a directed square lattice in which the sites are the points in the t, y plane with integer coordinates such that $t \geqslant 0$ and $t+y$ is even. The site (t, y) has two outwardly directed bonds leading to the nearest-neighbour sites $(t+1, y\pm 1)$, and the positive t axis, which we shall suppose to be horizontal, will be known as the 'preferred' direction. Dhar (1983) and Domany and Kinzel (1984) have shown that bond and site percolation on this directed square lattice are both cases of a one-dimensional stochastic cellular automaton model in which t is the time.

The cells of the automaton are labelled by the integer variable y and can be in one of two states 0 and 1. The state of each cell at $t=0$ is given and for $t \geqslant 1$ the state of the cell y at time t is determined, using probabilistic rules which are the same for each cell, by the states of the cells $y\pm 1$ at time $t-1$. The rules are embodied in the four conditional probabilities $P(1|0,0)=0$, $P(1|1,1)=p_2$, $P(1|1,0)=p_{\mathrm{u}}$, $P(1|0,1)=p_{\mathrm{d}}$ where the first index is the state of site y and the other two are the states of sites $y-1$ and $y+1$ respectively. The probabilities for y to be in state 0 are the complements of these. If at $t=0$ the cells with odd values of y are in state zero then the spacetime points for which $t+y$ is odd will correspond to a cell in state zero (i.e. odd cells at even times and even cells at odd times are known to be in state zero) since two zeros never give rise to a 1. We shall always suppose that this is the case and hence only sites on the spacetime lattice for which $t+y$ is even need be considered.

For the choice $p_u = p_d = p_2 = p$ the spacetime cells in state 1 correspond to directed square site percolation clusters which are attached to the y axis and when $p_u = p_d = p$ and $p_2 = 2p - p^2$, bond percolation clusters result. In the case $p_2 = 1$ the spacetime cells in state 1 form *compact* clusters on the square lattice (figure 1) since two cells in state 1 are always followed by a cell in state 1 and hence no holes can form. In this paper we shall consider only the case $p_2 = 1$ and only initial states in which all the cells in state 1 are contiguous. If there are m of these they will be said to form a seed of width m. Domany and Kinzel (1984) showed that for isotropic compact clusters ($p_u = p_d = p$) with a seed of width one there is a critical probability, $p_c = \frac{1}{2}$, above which there is a positive probability that at least one cell will be in state 1 at any time. The latter event corresponds to a compact cluster of infinite extent in the direction of the t axis. They also determined the probability $r_t(p)$ that at least one cell will be in state 1 at time t but that there are no such cells at time $t+1$. This event corresponds to a compact cluster of length $t+1$, where we define *cluster length* to be the number of atoms in a path from the seed to the terminal point. Notice that all compact clusters generated by the above process are either infinite or terminate after a finite number of steps in a single vertex. These clusters are only a subset of the compact directed animals considered by Bhat *et al* (1986, 1988) who allow two adjacent occupied sites ($y \pm 1$ say) at time $t-1$ to be followed by a vacant site y at time t. Their compactness condition is that all occupied sites in a given column t must be contiguous.

Here (§ 2) we rederive the results of Domany and Kinzel for compact clusters and also obtain the moment generating function and all the moments of the cluster length distribution. In § 3 a different technique is used to generalise the results to the anisotropic case $p_u \neq p_d$ and to seeds of arbitrary width m. The critical curve for the anisotropic case is found to be $p_u + p_d = 1$ and a duality relation connecting points on either side of this curve is also discussed in this section. In § 4 the same method is used to investigate the cluster size distribution which is the more usual function of interest in percolation theory. A recurrence relation for the moment generating function of this distribution is obtained which in principle allows all the moments to be determined. In fact we have obtained only the first three moments, the critical exponents of which are consistent with the existence of a scaling size. Hyperscaling, which is found to be valid numerically for directed bond and site percolation (De'Bell and Essam 1983), is found not to hold for compact clusters. This is perhaps not surprising

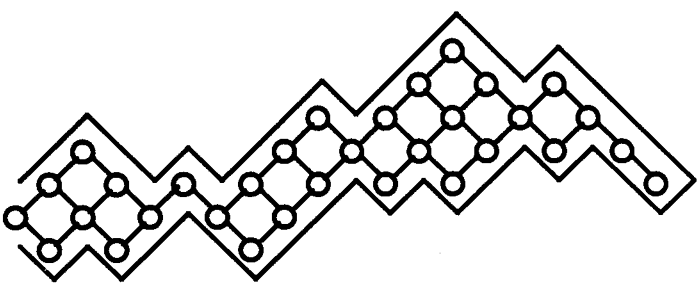

Figure 1. A directed compact cluster with 19 growth stages, length 20 and size 32 together with the corresponding pair of parallel walks of 20 steps.

since the usual arguments for hyperscaling breakdown for these clusters. Further results of this work are summarised and discussed in § 5.

2. Isotropic clusters grown from a seed of width one

We begin by considering the isotropic case $p_u = p_d = p$. It was noted by Domany and Kinzel (1984) that compact percolation clusters grown from a seed of width one may be placed in one-to-one correspondence with pairs of directed parallel walks on the dual lattice which intersect only on the last step (see figure 1). One of the walks follows the upper edge of the cluster as closely as possible and the other follows the lower edge. A step which corresponds to an increase in the cluster width will be said to be outward. For a cluster which terminates after t growth stages (and therefore has length $L = t+1$) the walks will meet after each one has executed $t+1$ steps. The number of pairs of walks of the above type with a given length has been enumerated by Delest and Viennot (1984) and their result may be taken over to give the following formula for the number of compact clusters w_t having exactly t growth stages:

$$w_t = \frac{1}{t+2}\binom{2t+2}{t+1}. \tag{2.1}$$

This has generating function

$$W(x) = \sum_{t=0}^{\infty} w_t x^t = [1-2x-(1-4x)^{1/2}]/2x^2. \tag{2.2}$$

The probability $r_t(p)$ that a cluster grown for a single occupied site will terminate after t growth stages will be known as the cluster length distribution since such a cluster is defined to have length $L = t+1$. The probability of occurrence of a specific cluster having exactly t growth stages is $p^t q^{t+2}$ since a factor p is associated with each outward step and a factor $q(=1-p)$ with each inward step of the corresponding walk, and there are two more inward steps than outward steps and hence

$$r_t(p) = w_t p^t q^{t+2} \tag{2.3}$$

and using Stirling's formula for the factorials in w_t we obtain, in agreement with Domany and Kinzel (1984), for $p \to p_c = \frac{1}{2}$:

$$r_t(p) \cong \frac{1}{\sqrt{\pi} t^{3/2}} \exp(-t/\xi_{\Uparrow}) \tag{2.4}$$

where $\xi_{\Uparrow}(p)$ is the decay length:

$$\xi_{\Uparrow}(p) = \frac{1}{(1-2p)^2} \tag{2.5}$$

and hence the critical exponent $\nu_{\Uparrow} = 2$.

The probability $Q(p)$ that a cluster of the above type is finite is given by

$$Q(p) = \sum_{t=0}^{\infty} r_t(p) = q^2 W(pq). \tag{2.6}$$

The argument of the square root in (2.2) with $x=pq$ evaluates to $(1-2p)^2$ and the choice of its sign depends on whether p is above or below p_c, thus

$$Q(p)=\begin{cases}1 & p<p_c \\ (q/p)^2 & p\geqslant p_c\end{cases} \tag{2.7}$$

from which we obtain the percolation probability

$$P(p)=1-Q(p)=(2p-1)/p^2 \tag{2.8}$$

for $p>p_c$ and hence the critical exponent $\beta=1$.

The kth moment of the probability distribution $r_t(p)$ is defined by

$$\mu_k(p)=\sum_{t=0}^{\infty} t^k r_t(p) \tag{2.9}$$

and may be obtained from the generating function

$$R(p,z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\mu_k(p)z^k=\sum_{t=0}^{\infty}\mathrm{e}^{-zt}r_t(p)=q^2W(pq\mathrm{e}^{-z}). \tag{2.10}$$

Clearly $\mu_0(p)=R(p,0)=Q(p)$ and using (2.10) we find that the expected length $L(p)$ given that the cluster is finite is

$$L(p)=1+\mu_1(p)/\mu_0(p)=\begin{cases}1/(1-2p) & \text{for } p<p_c \\ 1/(1-2q) & \text{for } p>p_c\end{cases} \tag{2.11}$$

$$=L(1-p). \tag{2.12}$$

Notice that $L(p)$ has critical exponent $\tau=1$ which is different from that of the decay length. In general we expect, from scaling theory, that $\tau+\beta=\nu_{\Uparrow}$. The symmetry of $L(p)$ about p_c arises from the fact that the normalised generating function, $R(p,z)/q^2$, is invariant under interchange of p and q and that $L(p)$ is determined by a ratio of moments. This 'duality' is therefore a property shared by all moments when normalised by division by μ_0 (see § 3.3 for discussion of the use of the term 'duality').

For $k\geqslant 1$ we find from (2.10), expanding $W(pq\mathrm{e}^{-z})$ in powers of z with the aid of the exact expression (2.2) and keeping the dominant term as $p\to p_c$,

$$\mu_k(p)\cong\frac{(2k-2)!}{(k-1)!4^{k-1}|1-2p|^{2k-1}} \qquad \text{for } p\to p_c$$

$$\sim \xi_{\Uparrow}(p)^{k-1/2}. \tag{2.13}$$

Before going on to consider the cluster size distribution we first rederive the above results by a more general technique and at the same time obtain the extension to the anisotropic problem $p_u\neq p_d$ and to clusters based on a seed of width m.

3. Anisotropic clusters grown from a seed of width m

Let $R_m(p,z)$ be the moment generating function as defined in (2.10) but where now $r_t(p)$ is replaced by $r_t(p,m)$, the probability that a cluster grown from a seed of width m will terminate after t growth stages, and p denotes the pair of variables $\{p_u,p_d\}$.

3.1. Recurrence relations

At $t=1$ the cluster has three possible widths, $m-1$, m and $m+1$ which occur with probabilities $d=q_u q_d$, $p_u q_d+q_u p_d$ and $c=p_u p_d$ respectively. This leads to the recurrence relation for $m \geqslant 2$ and $t \geqslant 1$

$$r_t(p, m)=c r_{t-1}(p, m+1)+(1-c-d) r_{t-1}(p, m)+d r_{t-1}(p, m-1) \tag{3.1}$$

with initial condition

$$r_0(p, m)=\begin{cases} d & \text{for } m=1 \\ 0 & \text{for } m \geqslant 2. \end{cases} \tag{3.2}$$

The corresponding relation for the moment generating function is, for $m \geqslant 2$,

$$R_m(p, z)=\mathrm{e}^{-z}[c R_{m+1}(p, z)+(1-c-d) R_m(p, z)+d R_{m-1}(p, z)]. \tag{3.3}$$

In the case $m=1$ and $t \geqslant 1$

$$r_t(p, 1)=c r_{t-1}(p, 2)+(1-c-d) r_{t-1}(p, 1) \tag{3.4}$$

which gives rise to

$$R_1(p, z)-r_0(p, 1)=\mathrm{e}^{-z}[c R_2(p, z)+(1-c-d) R_1(p, z)] \tag{3.5}$$

so that (3.3) is valid with $m=1$ provided that

$$R_0(p, z)=\mathrm{e}^{z}. \tag{3.6}$$

3.2. Solution of the recurrence relation for the moment generating function

Equation (3.3) has solutions of the form λ^m where λ is a root of

$$c\lambda^2+(1-c-d)-\mathrm{e}^{z})\lambda+d=0 \tag{3.7}$$

and the solution which remains bounded as $z \rightarrow \infty$ is

$$\lambda=\{c+d+\mathrm{e}^{z}-1-\sqrt{[(1-c-d-\mathrm{e}^{z})^2-4cd]}\}/2c. \tag{3.8}$$

Imposing the additional condition (3.6) gives

$$R_m(p, z)=\mathrm{e}^{z}\lambda(z)^m \tag{3.9}$$

which reduces to $R(p, z)$ in the case $m=1$ and $p_u=p_d=p$. Setting $z=0$ gives the probability $Q_m(p)$ that a cluster grown from a seed of width m is infinite:

$$Q_m(p)=\begin{cases} 1 & \text{for } c<d \\ (d/c)^m & \text{for } c \geqslant d. \end{cases} \tag{3.10}$$

The critical curve $c=d$ has equation

$$p_u+p_d=1 \tag{3.11}$$

and the asymptotic form of the percolation probability near this curve, generalised to a seed of width m, is

$$P_m(p) \cong m(p_u+p_d-1)/p_u p_d \tag{3.12}$$

and hence the critical exponent $\beta=1$ for all m and all points on the curve, as expected. Expanding the generating function in powers of z to obtain the moments shows that the mean length is

$$L_m(p)=\begin{cases} m/(1-p_u-p_d) & \text{for } p_u+p_d<1 \\ m/(1-q_u-q_d) & \text{for } p_u+p_d>1 \end{cases} \tag{3.13}$$

and that the critical exponent of the kth moment, for all m and all points on the critical curve, is $2k-1$. This critical behaviour of the moments and that of the percolation probability also follows from the *scaling form*

$$\lambda(z) \cong 1+\{(d-c)-|d-c|\surd[1+4cz/(c-d)^2]\}$$

by which we mean

$$\lim_{d\to c}[(\lambda(z)-1)/(d-c)] = 1-\mathrm{sgn}(d-c)\surd(1+4cZ)$$

where the limit is taken with $Z=z/(c-d)^2$ fixed.

3.3. Duality

Notice that $R_m(p, z)/d^m$ is invariant under interchange of c and d, which is an extension of the duality property introduced in the previous section. It follows that all moments when divided by d^m have this property which manifests itself in the symmetry of the expressions for $Q_m(p)$ and $L_m(p)$ given above. This result will now be obtained by a direct argument which extends (2.3).

Let c_m be a possible finite cluster grown from a seed of width m. The probability of occurrence of such a cluster may be obtained by considering the steps in the bounding walks on the dual (figure 1). Now P_u is the probability that the upper walk moves outwards at a given step and q_u is the probability that it moves inwards. The corresponding probabilities for the lower walk are p_d and q_d respectively. The smallest possible cluster is generated by taking all inward steps and terminates after each walk has made m such steps; it therefore occurs with probability $(q_u q_d)^m$. Any other cluster will have in addition a number of outward steps some of which may be up and the others will be down. In order for the cluster to terminate there must be an inward step of the lower boundary for every outward step of the upper boundary and vice versa. The probability that c_m occurs is therefore:

$$Pr(c_m) = (q_u q_d)^m (p_u q_d)^u (p_d q_u)^d \tag{3.14}$$

where u is the number of upward steps of the upper bounding walk and d is the number of downward steps of the lower bounding walk of c_m. It follows that $\mathrm{Pr}(c_m)/d^m$ is invariant under the simultaneous interchanges

$$p_u \longleftrightarrow q_d \qquad p_d \longleftrightarrow q_u. \tag{3.15}$$

The probability $r_t(p, m)$ is obtained summing (3.14) over all clusters for which $u+d=t$. hence $r_t(p, m)/d^m$ is invariant under the interchanges (3.15) and since from the recurrence relation (3.1) it may be written as a function of c and d only, it is invariant under interchange of c and d. Use of the definition of $R_m(p, z)$ in terms of $r_t(p, m)$ gives the observed duality property of this moment generating function.

We have called the symmetry relation 'duality' for the following reasons. Returning to the automaton model described in the introduction we note that $P(0|0, 0)=1$ and with the initial condition under consideration (a single seed of width m) the cells in state 0 are contiguous but fall into two groups, those above the seed and those below. However, suppose we consider the model to be a limiting case of a finite circular automaton in which the cells are positioned on a circle at angles $\theta=2\pi y/N$ with $y=0, \ldots, N-1$, where N is an even integer, and again suppose that initially there are m cells in state 1 and that these cells have even y and are contiguous (i.e. there are no intervening even cells in state 0). The cells in state zero are now also contiguous and both types of cell remain contiguous for all t. The spacetime lattice is now on a cylinder and the cells of the spacetime lattice consist of a compact cluster of type 0 and a compact cluster of type 1 and the relation between them is symmetric. The cluster edges will still be referred to as upper and lower and passage through the cluster from lower to upper edge is in the anticlockwise direction. The bounding walks on the dual now separate the 1-cluster from the 0-cluster and the walk which bounds the upper edge of the 1-cluster bounds the lower edge of the 0-cluster. Hence outward moves for the 1-cluster are inward moves for the 0-cluster and vice versa (similarly for the other bounding walk). The properties of the 0-cluster, e.g. mean length and mean size, can therefore be obtained from those of the 1-cluster by the interchange (3.15) since the q are now the outward probabilities for the 0-cluster and u and d are interchanged because of the above correspondence between upper and lower edges.

The situation described in the previous paragraph is reminiscent of the well known two-dimensional Ising model in which clusters of up spins are separated from clusters of down spins by polygons on the dual. The symmetry property of the partition function (Kramers and Wannier 1941) in this case is less clear but can be obtained from matching the low-temperature cluster expansion with the high-temperature polygon expansion on the dual.

In percolation theory there are two types of coexisting cluster and for site percolation on the triangular lattice the clusters of one type fill the holes in the other type leading to $p_c = \frac{1}{2}$ (Sykes and Essam 1964). This property of site percolation clusters on the triangular lattice was known as a matching relation but may also be seen as a duality relation on a modified lattice (Essam 1979). In the compact cluster model considered here there is only one cluster of each type but if the initial state consisted of many seeds then the resulting set of clusters would be such that clusters of type 0 fill the holes in the clusters of type 1.

4. The cluster size distribution

The probability $p_s(p)$ that a cluster grown from a single site has s sites (i.e. size s) is one of the basic functions normally studied in percolation theory and we now consider its generalisation, $p_s(p, m)$, to clusters based on a seed of width m. The analysis follows closely that of the distribution $r_l(p, m)$ in the previous section but we have been unable to obtain an explicit solution of the recurrence relation satisfied by the moment generating function. However, closed formulae have been obtained for the mean cluster size and the second moment of the distribution. In principle such an expression could be obtained for any moment but the results become increasingly complex the higher the moment.

4.1. Recurrence relations

For $m \geq 2$ and $s \geq m+1$ the cluster size distribution satisfies the relation

$$p_s(p, m) = cp_{s-m}(p, m+1) + (1-c-d)p_{s-m}(p, m) + dp_{s-m}(p, m-1) \tag{4.1}$$

which is similar to equation (3.1) for the length distribution function except that $s-1$ is replaced by $s-m$ on the right-hand side. In the case $m = 1$ the last term is zero for

$s \geq 2$ and the boundary conditions are

$$p_s(p, m) = \begin{cases} d & \text{for } m = s = 1 \\ 0 & \text{for } m \geq 2 \text{ and } s \leq m. \end{cases} \tag{4.2}$$

The moment generating function may be defined by

$$G_m(p, h) = \sum_{s=m}^{\infty} e^{-sh} p_s(p, m) \tag{4.3}$$

and for all $m \geq 1$ it satisfies the relation

$$G_m(p, h) = e^{-mh}[cG_{m+1}(p, h) + (1 - c - d)G_m(p, h) + dG_{m-1}(p, h)] \tag{4.4}$$

provided that we define $G_0(p, h) = 1$. Setting $h = 0$ gives the probability that the cluster is finite:

$$G_m(p, 0) = Q_m(p) \tag{4.5}$$

an explicit formula for which is given by equation (3.10).

4.2. Duality

The probability $p_s(p, m)$ may be obtained by summing $Pr(c_m)$ over all clusters with s sites and hence $p_s(p, m)/d^m$ has the same duality property as $r_t(p, m)/d^m$ and so does $G_m(p, h)/d^m$. It is therefore only necessary to solve the recurrence relation (4.4) in the non-percolating region $c < d$ in order to obtain the complete solution. We have not so far obtained such a solution but all the usual critical exponents may be obtained from the first and second moments which we now derive. These moments when normalised by dividing by the zeroth moment will be invariant under the duality transformation $c \longleftrightarrow d$.

4.3. The moments

Firstly note that $Q_m(p)$ satisfies a second-order linear recurrence relation with constant coefficients:

$$cQ_{m+2}(p) - (c + d)Q_{m+1}(p) + dQ_m(p) = 0 \tag{4.6}$$

or

$$d\Delta(a^{-m}\Delta(a^m Q_m(p))) = 0 \tag{4.7}$$

where Δ is the usual forward difference operator and

$$a = c/d. \tag{4.8}$$

Equation (4.7) has the general solution

$$Q_m(p) = f_1(a) + f_2(a)a^{-m}. \tag{4.9}$$

For $c < d$ the second solution is invalid since it becomes unbounded as $m \to \infty$, hence $f_2(a) = 0$ and since $Q_0(p) = 1$ it follows that $Q_m(p) = 1$ for all m. For $c > d$, considering the smallest cluster for given m shows that $Q_m(p) \cong d^m \cong a^{-m}$ as $a \to 0$ and hence $f_1(a) = 0$, $f_2(a) = 1$ in agreement with (3.10).

The first moment of $p_s(p, m)$ when normalised by $Q_m(p)$ is the expected cluster size $S_m(p)$ given that the cluster is finite. Differentiating (4.4) with respect to h and setting $h=0$ gives, for $n \geqslant 1$ and $c<d$:

$$S_m(p) = cS_{m+1}(p) + (1-c-d)S_m(p) + dS_{m-1}(p) + m \tag{4.10}$$

which has an obvious direct interpretation and may be written

$$d\Delta(a^{-m}\Delta(a^m S_m(p))) = -m-1. \tag{4.11}$$

This relation has the general solution

$$S_m(p) = \frac{1}{2d}\left(\frac{m(1+a)}{(1-a)^2} + \frac{m^2}{1-a}\right) + g_1(a) + g_2(a)a^{-m}. \tag{4.12}$$

The last term is zero since it becomes unbounded as $m \to \infty$ and imposing the initial condition $S_0(p) = 0$ gives

$$S_m(p) = \frac{1}{2}\left(\frac{m(d+c)}{(d-c)^2} + \frac{m^2}{d-c}\right) \tag{4.13}$$

and hence the critical exponent $\gamma = 2$ all along the critical curve $a = 1$ (or $c = d$).

In order to find the gap exponent Δ we consider the second normalised moment $U_m(p)$ of the cluster size distribution. For $c<d$, differentiating (4.4) twice with respect to h, $U_m(p)$ may be seen to satisfy the relation

$$U_m(p) = cU_{m+1}(p) + (1-c-d)U_m(p) + dU_{m-1}(p) - m^2 + 2mS_m(p) \tag{4.14}$$

which may be written

$$d\Delta(a^{-m}\Delta(a^m U_m(p))) = (1-A)(m+1)^2 - B(m+1)^3 \tag{4.15}$$

where

$$A = (c+d)/(d-c)^2 \text{ and } B = 1/(d-c). \tag{4.16}$$

Imposing the same initial and boundary conditions as for $S_m(p)$, the required solution is

$$dU_m(p) = (A+B-1)t_1 + (3A+7B-3)t_2 + (A+6B-1)t_3 + Bt_4 \tag{4.17}$$

where

$$t_k = \frac{(k-1)!}{1-a} \sum_{n=1}^{k} {}^mC_n[a/(1-a)]^{k-n}. \tag{4.18}$$

The binomial coefficient ${}^mC_n$ is given the value zero for $n>m$. In the case $m=1$ only the term $n=1$ for each value of k is required and this yields the result

$$U_1(p) = [1-2b+2b^3-b^4+c(9-15b+3b^2+3b^3)+c^2(18-16b-2b^2)+10c^3]/(1-b)^5 \tag{4.19}$$

where $b = p_u + p_d$. Thus the second moment of the cluster size distribution has critical exponent 5 and since the first moment was found to have critical exponent 2 it follows that the gap exponent $\Delta = 3$ in agreement with the scaling relation $\Delta = \beta + \gamma$ together with the previous results $\beta = 1$, $\gamma = 2$. Inspection of (4.17) shows that Δ is independent of m.

5. Discussion

The length and size distributions of compact directed clusters grown from a seed of width m have been investigated. In the case of the length distribution an explicit formula for the moment generating function has been given and the critical exponent for the kth moment was found to be $2k-1$. The 'gap exponent' Δ_t for this distribution therefore has the value 2 in agreement with the result $\Delta_t = \nu_{\Uparrow}$ which results from a scaling hypothesis with characteristic length $\xi_{\Uparrow}(p)$. A recurrence relation has been given for the moment generating function of the cluster size distribution which is similar to that for the length distribution but has not been solved explicitly. However, the first three moments (zeroth, first and second) have been obtained and are consistent with a scaling hypothesis having characteristic size with critical ('gap') exponent $\Delta = 3$.

Domany and Kinzel (1984) have shown that the scaling width $\xi_{\perp}(p)$ has critical exponent $\nu_{\perp} = 1$. They deduced this result from the solution of a triangular lattice Ising model with three-spin interactions in alternate triangles (Verhagen 1976). The same result follows by considering the mean width of a cluster in the percolating phase as a function of its length. Using the idea of Domany and Kinzel (1984) that the upper and lower edges of the cluster are random walks with probabilities p_u and p_d of moving outwards respectively we see that the mean width is $2(p_u+p_d-1)t+m$ and hence (Kinzel 1983)

$$\xi_{\perp}^{(p)}/\xi_{\Uparrow}^{(p)} \sim p_u + p_d - 1 \tag{5.1}$$

and hence $\nu_{\perp} = \nu_{\Uparrow} - 1 = 1$. This result when combined with the results $\beta = 1$, $\Delta = 3$, $\nu_{\Uparrow} = 2$ leads to an inconsistency with the hyperscaling relation (Cardy and Sugar 1980, De'Bell and Essam 1983, Kinzel 1983)

$$\Delta + \beta = \nu_{\Uparrow} + D\nu_{\perp} \tag{5.2}$$

which is satisfied by series expansion estimates for ordinary directed percolation (De'Bell and Essam 1983). Here the number of perpendicular dimensions $D = 1$. This breakdown of hyperscaling is perhaps not surprising since the superlattice picture of an infinite cluster (Skal and Shklovskii 1975, Harms and Strayley 1982, Redner 1982) is clearly not valid for compact clusters. For finite compact clusters the scaling size is proportional to the volume $\xi_{\perp}(p)\xi_{\Uparrow}(p)^D$ which leads to the alternative scaling relation

$$\Delta = \nu_{\Uparrow} + D\nu_{\perp} \tag{5.3}$$

which is satisfied by our $D = 1$ results.

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